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Problem | Weight | Score
---|---|---
1 | 25 | 
2 | 25 | 
3 | 25 | 
4 | 25 | 
Total | 100 | 

Test Form A

INSTRUCTIONS

1. You have 2 hours to complete this exam.
2. This is a closed book exam. You may use one 8.5" × 11" note sheet.
3. Calculators are not allowed.
4. Solve each part of the problem in the space following the question. If you need more space, continue your solution on the reverse side labeling the page with the question number; for example, Problem 1.2 Continued. NO credit will be given to solutions that do not meet this requirement.
5. **DO NOT REMOVE ANY PAGES FROM THIS EXAM.** Loose papers will not be accepted and a grade of ZERO will be assigned.
6. The quality of your analysis and evaluation is as important as your answers. Your reasoning must be precise and clear; your complete English sentences should convey what you are doing. **To receive credit, you must show your work.**
Problem 1: (25 Points)

1. (5 points) A periodic signal \( f(t) \) is represented as

\[
f(t) = \sum_{n=0}^{\infty} \cos(\omega t) \cos(\omega n t)
\]

Determine the fundamental period \( T_0 \) of the signal \( f(t) \).

\[
f(t) = \cos(t) + \cos(2t) \cos(2t) + \cos(4t) \cos(4t)
\]

Using \( \cos x \cos y = \frac{1}{2} \cos(x+y) + \frac{1}{2} \cos(x-y) \)

\[
f(t) = \cos(t) + \frac{1}{2} \cos(3t) + \frac{1}{2} \cos(t) + \frac{1}{2} \cos(5t) + \frac{1}{2} \cos(3t)
\]

\[
f(t) = 2 \cos(t) + \frac{1}{2} \cos(3t) + \frac{1}{2} \cos(5t)
\]

To obtain \( f(t+\tau) = f(t) \), the following equalities must hold

\[
\tau = 2\pi n \quad \text{and} \quad \tau = 2m \pi \quad \text{and} \quad \tau = 5 \pi p
\]

where \( n, m \) and \( p \) are integers. Solving for

\[
\tau = 2\pi n = \frac{2\pi}{3} m = \frac{2\pi}{5} p
\]

The smallest integers satisfying equation (1) are \( n = 3, m = 3, p = 5 \).

It follows that

\[
T_0 = 2\pi
\]
2. (10 points) Figure 1 shows one period of a periodic signal \( f(t) \). Determine the trigonometric and complex exponential Fourier series representation of \( f(t) \) by specifying the values of \((a_0, a_n, b_n)\) and \(D_n\).

\[
\text{Slope } = \frac{1}{T/2} = \frac{1}{T_0}
\]

\[f(t) = \frac{2}{T_0} + \frac{BN}{T_0} \text{ for } -\frac{T_0}{2} < t < \frac{T_0}{2} \]

Figure 1: Periodic signal \( f(t) \).

Because \( f(t) = f(t + T) \) is an odd function, \( a_0 = 0 \), \( a_n = 0 \) for all \( n \), and

\[b_n = \frac{2}{T_0} \int_0^{T_0} f(t) \sin(n \omega_0 t) \, dt = \frac{2}{T_0} \int_0^{T_0} \frac{2}{T_0} t \sin \left( \frac{2 \pi n t}{T_0} \right) \, dt
\]

\[= \frac{8}{n \pi} \int_0^{T_0} t \sin \left( \frac{2 \pi n t}{T_0} \right) \, dt = \frac{8}{n \pi} \int_0^{T_0} \sin \left( \frac{2 \pi n t}{T_0} \right) \, dt
\]

\[= \frac{8}{n \pi} \left[ \frac{1}{2} \left( \sin \left( \frac{2 \pi n t}{T_0} \right) - \frac{2 \pi n t}{T_0} \cos \left( \frac{2 \pi n t}{T_0} \right) \right) \right]_0^{T_0}
\]

\[= \frac{8}{n \pi^2 \pi} \left[ \sin \left( \frac{2 \pi n T_0}{T_0} \right) - \pi n \cos \left( \frac{2 \pi n t}{T_0} \right) - 0 - 0 \right]
\]

\[b_n = -\frac{2}{\pi n} \cos(\pi n)
\]

\[D_0 = a_0 = 0
\]

\[D_n = \frac{1}{T_0} (a_n - j b_n) = \frac{1}{\pi n} \cos(\pi n) \text{ for } n = \pm 1, \pm 2, \ldots
\]

Note that \( D_n = 0 \) because \( f(t) \) is real-valued.

Because \( f(t) \) is real-valued and an odd function of time, \( D_n = 0 \) for \( n \).
3. (10 points) Figure 2 shows an M-file that first approximates a periodic function using a Fourier series representation with a limited number of harmonics, and then plots the approximation as a function of time. The period of the signal is four seconds.

\[ L = \{-10 : 0.01 : 10\} \]
\[ M = 20; \]
\[ N = 2; \]
\[ N = N + (4/pi/n)*sin(\pi/n) + \cos(\pi/n); \]
\[ plot(L, N); \]

Figure 2: MatLab code for approximating and plotting a periodic signal.

(a) (2 points) Specify the range of time \( t_1 \leq t \leq t_2 \) over which the function is approximated by providing the numeric values of \( t_1 \) and \( t_2 \).

From line (1), \( -10 \leq t \leq +10 \).

(b) (4 points) What is the DC value of the signal?

N represents the approximating and from line (2)

\[ N = 2 \] is the DC value.

(c) (4 points) What is the highest harmonic frequency, in units of Hertz, used in generating the Fourier series approximation?

From the same term in line (3)

\[ w_0n = \frac{n \pi \rho}{L} \]

The largest value of \( \rho \) is \( M = 20 \), and so

the highest harmonic is \( \frac{20 \pi}{L} = 10 \pi \frac{\text{rad}}{\text{sec}} \) or 51 Hz.
Problem 2: (25 points)

1. (9 points) Figure 3 shows the magnitude and phase spectra for a periodic signal y(t).

Figure 3: Magnitude and phase spectra of a periodic signal y(t).

(a) (3 points) Is the signal y(t) real, imaginary, or complex valued? Justify your answer.

From Figure 3, \( |D_n| = |D_{-n}| \) and \( XD_n = -XD_{-n} \), this is equivalent to \( D_n = D_{-n}^* \), and so \( y(t) \) is purely real.

(b) (3 points) Is the signal y(t) either an even or odd function of time? Justify your answer.

From Figure 3, using \( D_n = |D_n|e^{j\phi_n} \),

\[
D_1 = D_{-1} = -1 \\
D_2 = D_{-2} = -1 \\
D_0 = 0
\]

Because \( D_n = D_{-n} \), \( y(t) \) is an even function of time.

(c) (3 points) What is the DC value of y(t)?

The DC value of \( y(t) \) is \( D_0 = 2 \).
A real-valued periodic signal \( f(t) \) has the Fourier series representation

\[
f(t) = A + \sum_{n=1}^{\infty} 2|a_n| \cos(2\pi n \omega_0 t) \quad (\text{A})
\]

where \( A \) is a real-valued positive parameter. The signal \( f(t) \) is passed through an ideal lowpass filter with cutoff frequency \( \omega_c \) and amplitude \( B \), that is, the filter has the frequency response function

\[
H(\omega) = B \text{ rect} \left( \frac{\omega}{2\omega_c} \right)
\]

where the parameters \( B \) and \( \omega_c \) are real-valued positive constants. The output of the filter is

\[
g(t) = AB + 4 \cos(100t) \quad (\text{B})
\]

and the power of the signal \( g(t) \) is \( P_g = 24 \).

(a) (6 points) Specify the numeric value of the parameter \( B \).

(b) (6 points) Specify the numeric value of the parameter \( A \).

(c) (4 points) Specify the possible range of values of \( \omega_1 \), \( \omega_1 < \omega_2 < \omega_3 \), by providing numeric values for \( \omega_1 \) and \( \omega_2 \).

From equation (A), \( b_n^f = A_n \), \( b_n^f = 2 \ln 2 \), \( \omega_0 = 100 \).

From equation (D), \( b_n^g = AB_1 \), \( b_n^g = \frac{\omega}{\omega_0} = 2 \) \( (\& b_n^j = \theta n = 0) \)

\[
b_n^g = \frac{\omega}{\omega_0} = 2 \quad (\& b_n^j = -\theta n = 0)
\]

Using

\[
\begin{align*}
b_n^g &= H(\omega_0) b_n^f \\
n = 0: \quad b_0^g &= H(\omega_0) b_0^f \
&\Rightarrow AB = 8 \quad \checkmark \\
n = 1: \quad b_1^g &= H(\omega_0 \cdot 1) b_1^f \
&\Rightarrow 2 = B \quad 2 \
&\Rightarrow B = 1
\end{align*}
\]

Because \( b_n^g = 0 \) for \( n > 1 \), \( H(\omega_0 n) = 0 \) when \( \omega_0 n > 100 \).

From the sketch below, \( 100 < \omega_0 < 200 \)

\[\begin{array}{c}
\begin{array}{c}
\text{The cutoff frequency } \omega_c \\
\text{must be greater than } 100 \text{ but less than } 2\omega_0 = 200 \text{ to block the harmonic of } f(t) \text{ at } 2\omega_0.
\end{array}
\end{array}\]

\[P_g = |b_0^g|^2 + 2 \sum_{n=1}^{\infty} |b_n^g|^2 = A^2 + 2|b_1^g|^2 = A^2 + 2|2H(\omega_0)| = A^2 + 8
\]

Because \( P_g = 24 \), \( A^2 = 16 \) or \( A = \pm 4 \)
Problem 3: (25 points)

1. (12 points) By direct integration, determine the Fourier transform of the signal

\[ f(t) = e^{-t} u(t - 1). \]

\[
F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt = \int_{1}^{\infty} e^{-t} e^{-j\omega t} dt = \int_{1}^{\infty} e^{-(j\omega + 1)t} dt \\
= \left. \frac{-1}{j\omega + 1} e^{-j\omega t - t} \right|_{1}^{\infty} = -\frac{1}{j\omega + 1} \left[ 0 - e^{j\omega} e^{-1} \right] \\
F(\omega) = \frac{-e^{-j\omega} e^{-1}}{j\omega + 1}
\]
2. (13 points) A signal $g(t)$ has the Fourier transform

$$G(\omega) = [4 \text{sinc}(2\omega) + 2 \text{sinc}(\omega)] e^{-2\omega}.$$

(a) (3 points) Is $g(t)$ either an even or odd function of time? Justify your answer without directly determining $g(t)$.

$$G(\omega)$$ is neither an even or odd function of $\omega$ because $e^{-2\omega}$ is not an even or odd function of $\omega$. And so $g(t)$ is neither even or odd.

(b) (3 points) Is $g(t)$ real, imaginary, or complex valued? Justify your answer without directly determining $g(t)$.

$$G(\omega) = \left[4 \text{sinc}(-2\omega) + 2 \text{sinc}(-\omega)\right] e^{2\omega} = \left[4 \text{sinc}(2\omega) + 2 \text{sinc}(\omega)\right] e^{-2\omega} = G(\omega).$$

Because $G(\omega) = G(-\omega)$, $g(t)$ is a real-valued function.

(c) (7 points) Using an appropriate transform pair and property from the tables provided, determine $g(t)$.

1. Use $\text{rect} \left( \frac{t}{\frac{1}{2}} \right) \leftrightarrow \frac{1}{2} \text{sinc} \left( \frac{\omega}{2} \right)$

2. Use $f(t - \tau) \leftrightarrow F(\omega) e^{-j\omega \tau}$

$t = 4$:

$$\text{rect} \left( \frac{t}{4} \right) \leftrightarrow 4 \text{sinc} (2\omega)$$

$t = 2$:

$$\text{rect} \left( \frac{t}{2} \right) \leftrightarrow 2 \text{sinc} (\omega)$$

with $\tau_0 = 2$

\[
\text{rect} \left( \frac{t - 2}{4} \right) + \text{rect} \left( \frac{t - 2}{2} \right) \leftrightarrow \left[4 \text{sinc}(2\omega) + 2 \text{sinc}(\omega)\right] e^{-j2\omega}
\]
Problem 4: (25 points)

1. (12 points) A common problem in over-the-air television signal transmission is multipath distortion of the received signal due to the transmitted signal bouncing off structures. Typically, a strong main signal arrives at some time and a weaker ghost signal arrives later. Let $s(t)$ represent the transmitted signal, while $K_m s(t-t_m)$ and $K_g s(t-t_g)$ represent the main and ghost signals respectively. For simplicity, assume that the parameters $K_m$, $K_g$, $t_m$, and $t_g$ are positive constants. The received signal $r(t)$ is

$$r(t) = K_m s(t-t_m) + K_g s(t-t_g).$$

(a) (8 points) Given that the input and output of the communication channel are $s(t)$ and $r(t)$, respectively, specify the transfer function $H_c(w)$ of the communication channel.

$$H_c(w) = \frac{R(w)}{S(w)} = \frac{K_m s(w)e^{-j\omega t_m} + K_g s(w)e^{-j\omega t_g}}{s(w)}.$$

(b) (2 points) What would be the transfer function $G(w)$ of an ideal equalization system that would compensate for the effects of multipath? That is, given the input $r(t)$, the equalization system produces the response $s(t)$.

Choose $G(w) = \frac{1}{H_c(w)}$.

(c) (2 points) Is the ideal equalization system realizable? Justify your answer.

The signal $r(t)$ feeding into the equalization system is

$$r(t) = K_m s(t-t_m) + K_g s(t-t_g).$$

The signal leaving the equalization system is $s(t)$, $t_m$ seconds before $s(t)$ is received. The equalization is noncausal and therefore not realizable.
2. Consider the signal \( m(t) = \text{sinc}(2t) \).

Suppose that this message signal is used to modulate the carrier signal \( c_c(t) = \cos(20\pi t) \) so that the transmitted signal \( y(t) \) is

\[
y(t) = m(t)c_c(t).
\]

(a) (7 points) Determine and neatly sketch the Fourier transform \( Y(\omega) \).

Using Table 4.1:

\[
m(\omega) = \text{sinc}(\omega) \quad \rightarrow \quad M(\omega) = \frac{\pi}{2} \text{rect} \left( \frac{\omega}{\pi} \right)
\]

\[
Y(\omega) = \frac{1}{\sqrt{2\pi}} M(\omega) \cos(20\pi t e^{j\omega}) = \frac{M(\omega + 20\pi)}{2} + \frac{M(\omega - 20\pi)}{2}
\]

(b) (6 points) Find the energy of \( y(t) \).

\[
E_y = \frac{1}{2\pi} \int_{-\infty}^{\infty} |Y(\omega)|^2 d\omega = \frac{1}{\pi} \int_{0}^{\infty} |Y(\omega)|^2 d\omega = \frac{1}{\pi} \left( \frac{\pi}{4} \right)^2 y
\]

\[
E_y = \frac{\pi}{4}
\]